A Fixed Points Approach to stability of the Pexider Equation

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Abstract

Using the fixed point theorem we establish the Hyers-Ulam-Rassias stability of the generalized
Pexider functional equation

$$\frac{1}{|K|} \sum_{k \in K} f(x + k \cdot y) = g(x) + h(y), \quad x, y \in E$$

from a normed space $E$ into a complete $\beta$-normed space $F$, where $K$ is a finite abelian subgroup
of the automorphism group of the group $(E, +)$.

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1 Introduction and Preliminaries

Under what condition does there exist a group homomorphism near an approximate group homo-
morphism? This question concerning the stability of group homomorphisms was posed by S. M.
Ulam [58]. In 1941, the Ulam’s problem for the case of approximately additive mappings was solved
theorem for additive mappings and in 1978 Th. M. Rassias [47] generalized the Hyers’ theorem for
linear mappings by considering an unbounded Cauchy difference. The result of Rassias’ theorem
has been generalized by J.M. Rassias [44] and later by Gavruta [18] who permitted the Cauchy dif-
fERENCE to be bounded by a general control function. Since then, the stability problems for several
functional equations have been extensively investigated (cf. [16], [19], [23], [24], [25], [26], [27], [32],
[11], [44], [45], [48], [49]).

Let $E$ be a real vector space and $F$ be a real Banach space. Let $K$ be a finite abelian subgroup
of $Aut(E)$ (the automorphism group of the group $(E, +)$). $|K|$ denotes the order of $K$. Writing
the action of $k \in K$ on $x \in E$ as $k \cdot x$, we will say that $(f, g, h) : E \to F$ is a solution of the generalized
Pexider functional equation, if
\[ \frac{1}{|K|} \sum_{k \in K} f(x + k \cdot y) = g(x) + h(y), \quad x, y \in E \] (1.1)
The generalized quadratic functional equation
\[ \frac{1}{|K|} \sum_{k \in K} f(x + k \cdot y) = f(x) + f(y), \quad x, y \in E \] (1.2)
and the generalized Jensen functional equation
\[ \frac{1}{|K|} \sum_{k \in K} f(x + k \cdot y) = f(x), \quad x, y \in E \] (1.3)
are particulars cases of equation (1.1).
The functional equations (1.1), (1.2) and (1.3) appeared in several works by H. Stetkær, see for example [55], [56] and [57]. We refer also to the recent studies by L. Radoslaw [50] and [51].
If we set \( K = \{ I, \sigma \} \), were \( I : E \to E \) denotes the identity function and \( \sigma \) denote an additive function of \( E \), such that \( \sigma(\sigma(x)) = x \), for all \( x \in E \) then equation (1.1) reduces to the Pexider functionals equations
\[ f(x + y) + f(x + \sigma(y)) = g(x) + h(y), \quad x, y \in E, \] (1.4)
\[ f(x + y) = g(x) + h(y), \quad x, y \in E, \quad (\sigma = I) \] (1.5)
\[ f(x + y) + f(x - y) = g(x) + h(y), \quad x, y \in E, \quad (\sigma = -I) \] (1.6)
Y. H. Lee and K. W. Jung [33] obtained the Hyers-Ulam-Rassias of the Pexider functional equation (1.5). Jung [27] and Jung and Sahoo [30] investigated the Hyers-Ulam-Rassias stability of equation (1.6). Belaid et al. have proved the Hyers-Ulam stability of equation (1.1) and the Hyers-Ulam-Rassias stability of the functional equations (1.2), (1.3), (see [1], [11], [12] and [34]).
Recently, Radoslaw [50] obtained the Hyers-Ulam-Rassias stability of equation (1.1). In 2003 L. Ćadariu and V. Radu [9] notice that a fixed point alternative method is very important for the solution of the Hyers-Ulam stability problem. Subsequently, this method was applied to investigate the Hyers-Ulam-Rassias stability for Jensen functional equation, as well as for the additive Cauchy functional equation [12] by considering a general control function \( \varphi(x, y) \), with suitable properties, using such an elegant idea, several authors applied the method to investigate the stability of some functional equations, see for example [3], [4], [5], [6], [31], [35], [43].
In this paper, we will apply the fixed point method as in [9] to prove the Hyers-Ulam-Rassias stability of the functional equations (1.1), for a large classe of functions from a vector space \( E \) into complete \( \beta \)-normed space \( F \).

Now, we recall one of fundamental results of fixed point theory.
Let \( X \) be a set. A function \( d : X \times X \to [0, +\infty] \) is called a generalized metric on \( X \) if \( d \) satisfies the following:
(1) \( d(x, y) = 0 \) if and only if \( x = y \);
(2) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);
(2) \( d(x, z) \leq d(x, y) + d(y, z) \) for all \( x, y, z \in X \).
Theorem 1.1. \[15\] Suppose we are given a complete generalized metric space \((X, d)\) and a strictly contractive mapping \(J : X \to X\), while the Lipschitz constant \(L < 1\). If there exists a nonnegative integer \(k\) such that \(d(J^k, x, J^{k+1} x) < +\infty\) for some \(x \in X\), then the following are true:

1. the sequence \(J^n x\) converges to a fixed point \(x^*\) of \(J\);
2. \(x^*\) is the unique fixed point of \(J\) in the set \(Y = \{y \in X : d(J^k, x, y) < +\infty\}\);
3. \(d(y, x^*) \leq \frac{1}{1-L} d(y, Jy)\) for all \(y \in Y\).

Throughout this paper, we fix a real number \(\beta\) with \(0 < \beta \leq 1\) and let \(\mathbb{K}\) denote either \(\mathbb{R}\) or \(\mathbb{C}\). Suppose \(E\) is a vector space over \(\mathbb{K}\). A function \(\| \cdot \|_\beta : E \to [0, \infty)\) is called a \(\beta\)-norm if and only if it satisfies

1. \(\|x\|_\beta = 0\), if and only if \(x = 0\);
2. \(\|\lambda x\|_\beta = |\lambda| \|x\|_\beta\) for all \(\lambda \in \mathbb{K}\) and all \(x \in E\);
3. \(\|x + y\|_\beta \leq \|x\|_\beta + \|y\|_\beta\) for all \(x, y \in E\).

2 main results

In the following theorem, by using an idea of Cădariu and Radu \[9, 12\], we prove the Hyers-Ulam-Rassias stability of the generalized Pexider functional equation (1.1).

Theorem 2.1. Let \(E\) be a vector space over \(\mathbb{K}\) and let \(F\) be a complete \(\beta\)-normed space over \(\mathbb{K}\). Let \(K\) be a finite abelian subgroup of the automorphism group of \((E, +)\). Let \(f : E \to F\) be a mapping for which there exists a function \(\varphi : E \times F \to [0, \infty)\) and a constant \(L < 1\), such that

\[
\frac{1}{|K|} \sum_{k \in K} f(x + k \cdot y) - g(x) - h(y) \leq \varphi(x, y) \tag{2.1}
\]

and

\[
\sum_{k \in K} \varphi(x + k \cdot x, y + k \cdot y) \leq (|K|)^\beta L \varphi(x, y) \tag{2.2}
\]

for all \(x, y \in E\). Then, there exists a unique solution \(q : E \to F\) of the generalized quadratic functional equation (1.2) and a unique solution \(j : E \to F\) of the generalized Jensen functional equation (1.3) such that

\[
\frac{1}{|K|} \sum_{k \in K} j(k \cdot x) = 0, \tag{2.3}
\]

\[
\|f(x) - q(x) - j(x) - g(0) - h(0)\|_\beta \leq \frac{2}{2\beta} \frac{1}{1-L} \chi(x, x) + \frac{1}{2\beta} \frac{1}{1-L} \psi(x, x), \tag{2.4}
\]

\[
\|g(x) - q(x) - j(x) - g(0)\|_\beta \leq \varphi(x, 0) + \frac{2}{2\beta} \frac{1}{1-L} \chi(x, x) + \frac{1}{2\beta} \frac{1}{1-L} \psi(x, x) \tag{2.5}
\]

and

\[
\|h(x) - q(x) - h(0)\|_\beta \leq \frac{1}{2\beta} \frac{1}{1-L} \psi(x, x) + \varphi(0, x) \tag{2.6}
\]

for all \(x \in E\), where

\[
\chi(x, y) = \frac{|K|}{|K|^\beta} \varphi(0, y) + \varphi(x, y) + \varphi(x, 0) + \varphi(0, y)
\]
\[+ \frac{1}{|K|^\beta} \sum_{k \in K} [\varphi(k \cdot x, y) + \varphi(k \cdot x, 0)] \]

and
\[
\psi(x, y) = \frac{|K|}{|K|^\beta} \varphi(0, y) + \frac{1}{|K|^\beta} \sum_{k \in K} [\varphi(k \cdot x, y) + \varphi(k \cdot x, 0)].
\]

Proof. Letting \( y = 0 \) in (2.1), to obtain
\[
\|f(x) - g(x) - h(0)\|_\beta \leq \varphi(x, 0)
\] (2.7)
for all \( x \in E \). By using (2.7), (2.1) and the triangle inequality, we get
\[
\| \frac{1}{|K|} \sum_{k \in K} f(x + k \cdot y) - f(x) - (h(y) - h(0)) \|_\beta \leq \frac{1}{|K|} \sum_{k \in K} f(x + k \cdot y) - g(x) - (h(y) - h(0)) \|_\beta
\] (2.8)
\[
+ \|g(x) - f(x) + h(0)\|_\beta \leq \varphi(x, y) + \varphi(x, 0)
\]
for all \( x, y \in E \). Replacing \( x \) by 0 in (2.1), we get
\[
\| \frac{1}{|K|} \sum_{k \in K} f(k \cdot y) - g(0) - h(y)\|_\beta \leq \varphi(0, y)
\] (2.9)
for all \( y \in E \). So inequalities (2.8), (2.9) and the triangle inequality implies that
\[
\| \frac{1}{|K|} \sum_{k \in K} f(x + k \cdot y) - f(x) - \frac{1}{|K|} \sum_{k \in K} f(k \cdot y) + g(0) + h(0)\|_\beta \leq \frac{1}{|K|} \sum_{k \in K} f(x + k \cdot y) - f(x) - (h(y) - h(0)) \|_\beta
\] (2.10)
\[
+ \| \frac{1}{|K|} \sum_{k \in K} f(k \cdot y) - h(0) - g(0)\|_\beta \leq \varphi(x, y) + \varphi(x, 0) + \varphi(0, y)
\]
for all \( x, y \in E \). Now, let
\[
\varphi(x) = \frac{1}{|K|} \sum_{k \in K} f(k \cdot x)
\] (2.11)
for all \( x \in E \). Then, \( \varphi \) satisfies
\[
\frac{1}{|K|} \sum_{k \in K} \varphi(k \cdot x) = \varphi(x)
\] (2.12)
for all \( x \in E \). Furthermore, in view of (2.10), (2.12) and the triangle inequality, we have
\[
\| \frac{1}{|K|} \sum_{k' \in K} [\varphi(x + k' \cdot y) - \varphi(x) - \varphi(y) + g(0) + h(0)] \|_\beta
\] (2.13)
\[
= \| \frac{1}{|K|} \sum_{k' \in K} \frac{1}{|K|} \sum_{k \in K} f(k \cdot x + k'k \cdot y) - \frac{1}{|K|} \sum_{k \in K} f(k \cdot x) - \frac{1}{|K|^2} \sum_{k,k' \in K} f(kk' \cdot y) + g(0) + h(0) \|_\beta
\]
\[
\leq \frac{1}{|K|^\beta} \sum_{k \in K} \| \frac{1}{|K|} \sum_{k' \in K} f(k \cdot x + k' \cdot y) - f(k \cdot x) - \frac{1}{|K|} \sum_{k' \in K} f(k' \cdot y) + g(0) + h(0) \|_\beta
\]
\[ \leq \frac{1}{|K|^\beta} \sum_{k \in K} [\varphi(k \cdot x, y) + \varphi(k \cdot x, 0)] + \frac{|K|}{|K|^\beta} \varphi(0, y) = \psi(x, y). \]

Since \( K \) is an abelian subgroup, so by using (2.2), we get
\[
\sum_{k \in K} \psi(x + k \cdot x, y + k \cdot y) \leq (2|K|)^\beta L\psi(x, y)
\] (2.14)

for all \( x, y \in E \). Let us consider the set \( X := \{ g : E \to F \} \) and introduce the generalized metric on \( X \) as follows:
\[
d(g, h) = \inf \{ C \in [0, \infty] : \| g(x) - h(x) \|_\beta \leq C\psi(x, x), \forall x \in E \}. \quad (2.15)
\]

Let \( f_n \) be a Cauchy sequence in \( (X, d) \). According to the definition of the Cauchy sequence, for any given \( \varepsilon > 0 \), there exists a positive integer \( N \) such that
\[
d(f_n, f_m) \leq \varepsilon
\] (2.16)

for all integer \( m, n \) such that \( m \geq N \) and \( n \geq N \). That is, by considering the definition of the generalized metric \( d \)
\[
\| f_m(x) - f_n(x) \|_\beta \leq \varepsilon\psi(x, x)
\] (2.17)

for all integer \( m, n \) such that \( m \geq N \) and \( n \geq N \), which implies that \( f_n(x) \) is a Cauchy sequence in \( F \), for any fixed \( x \in E \). Since \( F \) is complete, \( f_n(x) \) converges in \( F \) for each \( x \) in \( E \). Hence, we can define a function \( f : E \to F \) by
\[
f(x) = \lim_{n \to \infty} f_n(x).
\] (2.18)

As a similar proof to [34], we consider the linear operator \( J : X \to X \) such that
\[
(Jh)(x) = \frac{1}{2|K|} \sum_{k \in K} h(x + k \cdot x)
\] (2.19)

for all \( x \in E \). By induction, we can easily show that
\[
(J^n h)(x) = \frac{1}{(2|K|)^n} \sum_{k_1, \ldots, k_n \in K} h \left( x + \sum_{i_j < i_{j+1}, k_{i_j} \in \{k_1, \ldots, k_n\}} (k_{i_1} \ldots k_{i_p}) \cdot x \right)
\] (2.20)

for all integer \( n \).

First, we assert that \( J \) is strictly contractive on \( X \). Given \( g \) and \( h \) in \( X \), let \( C \in [0, \infty) \) be an arbitrary constant with \( d(g, h) \leq C \), that is,
\[
\| g(x) - h(x) \|_\beta \leq C\psi(x, x)
\] (2.21)
for all \( x \in E \). So, it follows from (2.19), (2.14) and (2.21) we get

\[
\|(Jg)(x) - (Jh)(x)\|_\beta = \frac{1}{2|K|} \sum_{k \in K} g(x + k \cdot x) - \frac{1}{2|K|} \sum_{k \in K} h(x + k \cdot x)\|_\beta
\]

\[
= \frac{1}{(2|K|)^\beta} \sum_{k \in K} g(x + k \cdot x) - h(x + k \cdot x)\|_\beta
\]

\[
\leq \frac{1}{(2|K|)^\beta} \sum_{k \in K} \|g(x + k \cdot x) - h(x + k \cdot x)\|_\beta
\]

\[
\leq \frac{1}{(2|K|)^\beta} C \sum_{k \in K} \psi(x + k \cdot x, x + k \cdot x)
\]

\[
\leq CL\psi(x, x)
\]

for all \( x \in E \), that is, \( d(Jg, Jh) \leq LC \). Hence, we conclude that

\[ d(Jg, Jh) \leq Ld(g, h) \]

for any \( g, h \in X \). Now, we claim that

\[ d(J(\varphi - g(0) - h(0)), \varphi - g(0) - h(0)) < \infty. \tag{2.22} \]

By letting \( y = x \) in (2.13), we obtain

\[
\|(J(\varphi-g(0)-h(0)))(x) - (\varphi-g(0)-h(0))(x)\|_\beta = \frac{1}{2^\beta} \||J| \sum_{k \in K} \varphi(x + k \cdot x) - 2\varphi(x) + g(0) + h(0)\|_\beta \leq \frac{1}{2^\beta} \psi(x, x) \tag{2.23} \]

for all \( x \in E \), that is

\[ d(J(\varphi - g(0) - h(0)), \varphi - g(0) - h(0)) \leq \frac{1}{2^\beta} < \infty \tag{2.24} \]

From Theorem 1.1, there exists a fixed point of \( J \) which is a function \( q : E \to F \) such that \( \lim_{n \to \infty} d(J^n(\varphi - g(0) - h(0)), q) = 0 \). Since \( d(J^n(\varphi - g(0) - h(0)), q) \to 0 \) as \( n \to \infty \), there exists a sequence \( \{C_n\} \) such that \( \lim_{n \to \infty} C_n = 0 \) and \( d(J^n\varphi - g(0) - h(0), q) \leq C_n \) for every \( n \in \mathbb{N} \). Hence, from the definition of \( d \), we get

\[ \|(J^n(\varphi - g(0) - h(0))(x) - q(x)\|_\beta \leq C_n\psi(x, x) \tag{2.25} \]

for all \( x \in E \). Therefore,

\[ \lim_{n \to \infty} \|(J^n(\varphi - g(0) - h(0))(x) - q(x)\|_\beta = 0, \tag{2.26} \]

for all \( x \in E \).

Now, if we put \( \kappa(x) = \varphi(x) - g(0) - h(0) \), by using induction on \( n \) we prove the validity of following inequality

\[ \frac{1}{|K|} \sum_{k \in K} J^n\kappa(x + k \cdot y) - J^n\kappa(x) - J^n\kappa(y)\|_\beta \leq L^n\psi(x, y). \tag{2.27} \]
In view of the commutativity of $K$ the inequalities \([2.13],\ (2.14)\) we have

\[
\left\| \frac{1}{|K|} \sum_{k \in K} J f(x + k \cdot y) - J \kappa(x) - J \kappa(y) \right\|_{\beta}
\]

\[= \left\| \frac{1}{|K|} \sum_{k \in K} 2\left|\frac{1}{K}\right| \sum_{k' \in K} \kappa(x + k \cdot y + k_1 \cdot x + k_1 \cdot y) - \frac{1}{2|K|} \sum_{k \in K} \kappa(x + k \cdot x) - \frac{1}{2|K|} \sum_{k \in K} \kappa(y + k \cdot y) \right\|_{\beta}
\]

\[\leq \left(2|K|\right)^{\beta} \sum_{k \in K} \left\| \frac{1}{|K|} \sum_{k' \in K} \kappa(x + k' \cdot x + k_1 \cdot x + k_1 \cdot y) - \kappa(x + k \cdot x) - \kappa(y + x \cdot y) \right\|_{\beta}
\]

\[\leq \left(2|K|\right)^{\beta} \sum_{k' \in K} \left\| \frac{1}{|K|} \sum_{k \in K} J^n \kappa(x + k' \cdot x + k \cdot y + k_1 \cdot y) - J^n \kappa(x + k' \cdot x) - J^n \kappa(y + k' \cdot y) \right\|_{\beta}
\]

\[\leq \left(2|K|\right)^{\beta} \sum_{k' \in K} \left\| \frac{1}{|K|} \sum_{k \in K} L^n \psi(x + k' \cdot x + k_1 \cdot y + k_1 \cdot y) \right\|_{\beta}
\]

which proves \([2.27]\) for $n + 1$. Now, by letting $n \to \infty$, in \([2.27]\), we obtain that $q$ is a solution of equation \((1.2)\). According to the fixed point theorem (Theorem 1.1, (3)) and inequality \((2.24)\), we get

\[d(\varphi - g(0) - h(0), q) \leq \frac{1}{1 - L} d(J(\varphi - g(0) - h(0)), \varphi - g(0) - h(0)) \leq \frac{1}{2^\beta (1 - L)} \]

and so we have

\[\|\varphi(x) - q(x) - g(0) - h(0)\| \leq \frac{1}{2^\beta (1 - L)} \psi(x,x) \]

for all $x \in E$. On the other hand if we put

\[
\omega(x) = f(x) - \varphi(x) = f(x) - \frac{1}{|K|} \sum_{k \in K} f(k \cdot x)
\]
for all $x \in E$, it follows from inequalities (2.10), (2.13) and the triangle inequality that

$$
\| \frac{1}{|K|} \sum_{k' \in K} \omega(x + k' \cdot y) - \omega(x) \|_\beta 
$$

(2.31)

$$
= \| \frac{1}{|K|} \sum_{k' \in K} f(x + k' \cdot y) - \frac{1}{|K|} \sum_{k \in K} \varphi(x + k \cdot y) - f(x) + \varphi(x) \|_\beta 
$$

$$
\leq \| \frac{1}{|K|} \sum_{k \in K} \varphi(x + k \cdot y) + \varphi(x) - g(0) - h(0) \|_\beta 
$$

$$
+ \| \frac{1}{|K|} \sum_{k' \in K} f(x + k' \cdot y) - f(x) - \frac{1}{|K|} \sum_{k' \in K} f(k' \cdot y) + g(0) + h(0) \|_\beta 
$$

$$
\leq \frac{1}{|K|^\beta} \sum_{k \in K} [\varphi(k \cdot x, y) + \varphi(k \cdot x, 0)] + \frac{|K|}{|K|^\beta} \varphi(0, y) + \varphi(x, y) + \varphi(x, 0) + \varphi(0, y) = \chi(x, y)
$$

for all $x, y \in E$. By using the same definition for $X$ as in the above proof, the generalized metric on $X$

$$
d(g, h) = \inf\{C \in [0, \infty] : \|g(x) - h(x)\|_\beta \leq C \chi(x, x), \forall x \in E\}. \quad (2.32)
$$

and some ideas of [34], we will prove that there exists a unique solution $j$ of equation (1.3) such that

$$
\| \omega(x) - j(x) \|_\beta \leq \frac{1}{1 - L} \chi(x, x) 
$$

(2.33)

for all $x \in E$.

First, from (2.2) we can easily verify that $\chi(x, y)$ satisfies

$$
\sum_{k \in K} \chi(x + k \cdot x, y + k \cdot y) \leq (2|K|)^\beta \chi(x, y)
$$

(2.34)

Let us consider the function $T : X \rightarrow X$ defined by

$$
(Th)(x) = \frac{1}{2|K|} \sum_{k \in K} h(x + k \cdot x)
$$

(2.35)

for all $x \in E$. Given $g, h \in X$ and $C \in [0, \infty]$ such that $d(g, h) \leq C$, so we get

$$
\| (Tg)(x) - (Th)(x) \|_\beta = \| \frac{1}{2|K|} \sum_{k \in K} g(x + k \cdot x) - \frac{1}{2|K|} \sum_{k \in K} h(x + k \cdot x) \|_\beta 
$$

$$
= \frac{1}{2|K|^\beta} \| \sum_{k \in K} [g(x + k \cdot x) - h(x + k \cdot x)] \|_\beta 
$$

$$
\leq \frac{1}{2|K|^\beta} \sum_{k \in K} \| g(x + k \cdot x) - h(x + k \cdot x) \|_\beta \leq C L \chi(x, x)
$$
for all $x \in E$. Hence, we see that $d(Tg, Th) \leq Ld(g, h)$ for all $g, h \in X$. So $T$ is a strictly contractive operator.

Putting $y = x$ in (2.31), we have

$$
\| \frac{1}{2K} \sum_{k \in K} \omega(x + k \cdot x) - \frac{1}{2} \omega(x) \|_\beta \leq \frac{1}{2\beta} \chi(x, x)
$$

(2.36)

for all $x \in E$, so by the triangle inequality, we get

$$
d(T\omega, \omega) \leq \frac{2}{2\beta}.
$$

(2.37)

From the fixed point theorem (Theorem 1.1), it follows that there exits a fixed point $j$ of $T$ in $X$ such that

$$
j(x) = \lim_{n \to \infty} \frac{1}{|2K|} \sum_{k_1, \ldots, k_n \in K} \omega \left( x + \sum_{i_j < i_{j+1}, k_{i_j} \in \{k_1, \ldots, k_n\}} [(k_{i_1}) \cdots (k_{i_p})] \cdot x \right)
$$

(2.38)

for all $x \in E$ and

$$
d(\omega, j) \leq \frac{1}{1-L} d(T\omega, \omega).
$$

(2.39)

So, it follows from (2.37) and (2.39) that

$$
\| \omega(x) - j(x) \|_\beta \leq \frac{2}{2\beta} \frac{1}{1-L} \chi(x, x)
$$

(2.40)

for all $x \in E$.

By the same reasoning as in the above proof, one can show by induction that

$$
\| \frac{1}{|K|} \sum_{k \in K} T^n \omega(x + k \cdot y) - T^n \omega(x) \|_\beta \leq L^n \chi(x, y)
$$

(2.41)

for all $x, y \in E$ and for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.41), we get that $j$ is a solution of the generalized Jensen functional equation (1.3).

From (2.11), (2.29) (2.30), (2.40) and the triangle inequality, we obtain

$$
\| f(x) - q(x) - j(x) - g(0) - h(0) \|_\beta \leq \frac{2}{2\beta} \frac{1}{1-L} \chi(x, x) + \frac{1}{2\beta} \frac{1}{1-L} \psi(x, x),
$$

(2.42)

and

$$
\| g(x) - q(x) - j(x) - g(0) \|_\beta \leq \varphi(x, 0) + \frac{2}{2\beta} \frac{1}{1-L} \chi(x, x) + \frac{1}{2\beta} \frac{1}{1-L} \psi(x, x)
$$

(2.43)

and

$$
\| h(x) - q(x) - h(0) \|_\beta \leq \frac{1}{2\beta} \frac{1}{1-L} \psi(x, x) + \varphi(0, x)
$$

(2.44)

for all $x \in E$.

Finally, in the following we will verify that the solution $j$ satisfies the condition

$$
\frac{1}{|K|} \sum_{k \in K} j(k \cdot x) = 0
$$

(2.45)
for all $x \in E$ and we will prove the uniqueness of the solutions $q$ and $j$ which satisfy the inequalities (2.42), (2.43) and (2.44).

Due to definition of $\omega$, we get $
 \frac{1}{|K|} \sum_{k \in K} \omega(k \cdot x) = 0$ for all $x \in E$, so we get $
 \frac{1}{|K|} \sum_{k \in K} T \omega(k \cdot x) = 0, \ldots, \nu \frac{1}{|K|} \sum_{k \in K} T^n \omega(k \cdot x) = 0$. So, by letting $n \to \infty$, we obtain the relation (2.45).

Now, according to (2.44) and (2.2) we get by induction that

$$\|J^n(h - h(0))(x) - q(x)\|_\beta \leq L^n[\frac{1}{2^\beta} \frac{1}{1 - L} \psi(x, x) + \varphi(0, x)]$$

(2.46)

for all $x \in E$ and for all $n \in \mathbb{N}$. So, by letting $n \to \infty$, we get

$$\lim_{n \to \infty} J^n(h - h(0))(x) = q(x)$$

(2.47)

for all $x \in E$, which proves uniqueness of $q$.

In a similar way, by induction we obtain

$$\|A^n(f - q - h(0) - g(0))(x) - j(x)\|_\beta \leq L^n[\frac{1}{1 - L} \chi(x, x) + \frac{1}{2^\beta} \frac{1}{1 - L} \psi(x, x)]$$

(2.48)

for all $x \in E$ and for all $n \in \mathbb{N}$, where

$$A l(x) = \frac{1}{|K|} \sum_{k \in K} l(x + k \cdot x).$$

Consequently, we have

$$\lim_{n \to \infty} A^n(f - q - h(0) - g(0))(x) = j(x)$$

(2.49)

for all $x \in E$. This proves the uniqueness of the function $j$ and this completes the proof of theorem.

In the following, we will investigate some special cases of Theorem 2.1, with the new weaker conditions.

**Corollary 2.2.** Let $E$ be a vector space over $\mathbb{K}$. Let $K$ be a finite abelian subgroup of the automorphism group of $(E, +)$. Let $\alpha = \frac{\log(|K|)}{\log(2)}$. Fix a nonnegative real number $\beta$ such that $\frac{\alpha}{\alpha + 1} < \beta < 1$ and choose a number $p$ with $0 < p < \beta + (\beta - 1)\alpha$ and let $F$ be a complete $\beta$-normed space over $\mathbb{K}$. If a function $f: E \to F$ satisfies

$$\| \frac{1}{|K|} \sum_{k \in K} f(x + k \cdot y) - g(x) - h(y)\|_\beta \leq \theta(||x||^p + ||y||^p)$$

(2.50)

and $\|x + k \cdot x\| \leq 2||x||$, for all $k \in K$, for all $x, y \in E$ and for some $\theta > 0$, then there exists a unique solution $q: E \to F$ of the generalized quadratic functional equation (1.2) and a unique solution $j: E \to F$ of the generalized Jensen functional equation (1.3) such that

$$\frac{1}{|K|} \sum_{k \in K} j(k \cdot x) = 0,$$

(2.51)
\begin{equation}
\|f(x) - q(x) - j(x) - g(0) - h(0)\|_\beta \leq \frac{\theta}{2^\beta} \frac{(2|K|)^\beta}{(2|K|)^\beta - 2^\beta |K|} \|x\|^p |\frac{|K|}{|K|^\beta} (4 + 4.3^p) + 2 + 2.3^p| \tag{2.52}
\end{equation}

\begin{equation}
\|g(x) - q(x) - j(x) - g(0)\|_\beta \leq \frac{\theta}{2^\beta} \frac{(2|K|)^\beta}{(2|K|)^\beta - 2^\beta |K|} \|x\|^p |\frac{|K|}{|K|^\beta} (4 + 43^p) + 2 + 2.3^p| + \theta \|x\|^p \tag{2.53}
\end{equation}

and

\begin{equation}
\|h(x) - q(x) - h(0)\|_\beta \leq \frac{\theta}{2^\beta} \frac{(2|K|)^\beta}{(2|K|)^\beta - 2^\beta |K|} \|x\|^p |\frac{|K|}{|K|^\beta} (2 + 2.3^p)| + \theta \|x\|^p \tag{2.54}
\end{equation}

for all \(x \in E\).

**Corollary 2.3.** Let \(E\) be a vector space over \(\mathbb{K}\). Fix a nonnegative real number \(\beta\) less than 1 and choose a number \(p\) with \(0 < p < 1\) and let \(F\) be a complete \(\beta\)-normed space over \(\mathbb{K}\). If a function \((f, g, h): E \rightarrow F\) satisfies

\begin{equation}
\|f(x + y) - g(x) - h(y)\|_\beta \leq \theta(\|x\|^p + \|y\|^p) \tag{2.55}
\end{equation}

for all \(x, y \in E\) and for some \(\theta > 0\), then there exists an unique additive function \(a: E \rightarrow F\) such that

\begin{equation}
\|f(x) - a(x) - g(0) - h(0)\|_\beta \leq \frac{\theta}{2^\beta} \frac{2^\beta}{2^\beta - 2p} \|x\|^p [6 + 6.3^p], \tag{2.56}
\end{equation}

\begin{equation}
\|g(x) - a(x) - g(0)\|_\beta \leq \frac{\theta}{2^\beta} \frac{2^\beta}{2^\beta - 2p} \|x\|^p [6 + 6.3^p] + \theta \|x\|^p \tag{2.57}
\end{equation}

and

\begin{equation}
\|h(x) - a(x) - h(0)\|_\beta \leq \frac{\theta}{2^\beta} \frac{2^\beta}{2^\beta - 2p} \|x\|^p [2 + 2.3^p] + \theta \|x\|^p \tag{2.58}
\end{equation}

for all \(x \in E\).

**Corollary 2.4.** Let \(E\) be a vector space over \(\mathbb{K}\). Let \(K = \{I, \sigma\}\) where \(\sigma\) is an involuntary of \(E\) \((\sigma(x + y) = \sigma(x) + \sigma(y)\) and \(\sigma(\sigma(x)) = x\) for all \(x, y \in E\)). Fix a nonnegative real number \(\beta\) such that \(\frac{1}{2} \leq \beta < 1\) and choose a number \(p\) with \(0 < p < 2\beta - 1\) and let \(F\) be a complete \(\beta\)-normed space over \(\mathbb{K}\). If a function \((f, g, h): E \rightarrow F\) satisfies

\begin{equation}
\|f(x + y) + f(x + \sigma(y)) - g(x) - h(y)\|_\beta \leq \theta(\|x\|^p + \|y\|^p) \tag{2.59}
\end{equation}

and \(\|x + \sigma(x)\| \leq 2\|x\|\), for all \(x, y \in E\) and for some \(\theta > 0\), then there exists a unique solution \(q: E \rightarrow F\) of the generalized quadratic functional equation

\begin{equation}
f(x + y) + f(x + \sigma(y)) = 2f(x) + 2f(y), \quad x, y \in E \tag{2.60}
\end{equation}

and a unique solution \(j: E \rightarrow F\) of the generalized Jensen functional equation

\begin{equation}
f(x + y) + f(x + \sigma(y)) = 2f(x), \quad x, y \in E \tag{2.61}
\end{equation}

such that

\begin{equation}
\sigma(x) = -j(x), \tag{2.62}
\end{equation}

\begin{equation}
\|f(x) - q(x) - j(x) - g(0) - h(0)\|_\beta \leq \theta \frac{4^\beta}{2^\beta |K|^\beta} \|x\|^p [\frac{2}{2^\beta} (4 + 4.3^p) + 2 + 2.3^p] \tag{2.63}
\end{equation}
\[ \|g(x) - q(x) - j(x) - g(0)\|_\beta \leq \frac{\theta}{2^3} \frac{4^\beta}{4^\beta - 2p_2} \|x\|^p \left[ \frac{2}{2^\beta} (4 + 43^p + 2 + 23^p) + \theta \|x\|^p \right] \] (2.64)

and

\[ \|h(x) - q(x) - h(0)\|_\beta \leq \frac{\theta}{2^3} \frac{4^\beta}{4^\beta - 2p_2} \|x\|^p \left[ \frac{2}{2^\beta} (2 + 23^p) + \theta \|x\|^p \right] \] (2.65)

for all \( x \in E \).

**Corollary 2.5.** Let \( E \) be a vector space over \( \mathbb{K} \) and let \( F \) be a complete \( \beta \)-normed space over \( \mathbb{K} \). Let \( f: E \rightarrow F \) be a mapping for which there exists a function \( \varphi : E \times F \rightarrow [0, \infty) \) and a constant \( L < 1 \), such that

\[ \|f(x + y) + f(x + \sigma(y)) - g(x) - h(y)\|_\beta \leq \varphi(x, y) \] (2.66)

and

\[ \varphi(2x, 2y) + \varphi(x + \sigma(x), y + \sigma(y)) \leq 4^\beta L \varphi(x, y) \] (2.67)

for all \( x, y \in E \). Then, there exists a unique solution \( q: E \rightarrow F \) of the generalized quadratic functional equation (2.62) and a unique solution \( j: E \rightarrow F \) of the generalized Jensen functional equation (2.63) such that

\[ j(\sigma(x)) = -j(x) \] (2.68)

\[ \|f(x) - q(x) - j(x) - g(0) - h(0)\|_\beta \leq 2 \frac{1}{2^\beta} \frac{1}{1 - L} \chi(x, x) + \frac{2}{2^\beta} \frac{1}{1 - L} \psi(x, x) \] (2.69)

\[ \|g(x) - q(x) - j(x) - g(0)\|_\beta \leq \varphi(x, 0) + \frac{2}{2^\beta} \frac{1}{1 - L} \chi(x, x) + \frac{1}{2^\beta} \frac{1}{1 - L} \psi(x, x) \] (2.70)

and

\[ \|h(x) - q(x) - h(0)\|_\beta \leq \frac{1}{2^\beta} \frac{1}{1 - L} \psi(x, x) + \varphi(0, x) \] (2.71)

for all \( x \in E \), where

\[ \chi(x, y) = \frac{2}{2^\beta} \varphi(0, y) + \varphi(x, y) + \varphi(x, 0) + \varphi(0, y) \]

\[ + \frac{1}{2^\beta} [\varphi(x, y) + \varphi(\sigma(x), y) + \varphi(x, 0) + \varphi(\sigma(x), 0)] \]

and

\[ \psi(x, y) = \frac{2}{2^\beta} \varphi(0, y) + \frac{1}{2^\beta} [\varphi(x, y) + \varphi(\sigma(x), y) + \varphi(x, 0) + \varphi(\sigma(x), 0)]. \]

**Corollary 2.6.** Let \( E \) be a vector space over \( \mathbb{K} \) and let \( F \) be a complete \( \beta \)-normed space over \( \mathbb{K} \). Let \( f: E \rightarrow F \) be a mapping for which there exists a function \( \varphi : E \times F \rightarrow [0, \infty) \) and a constant \( L < 1 \), such that

\[ \|f(x + y) - g(x) - h(y)\|_\beta \leq \varphi(x, y) \] (2.72)

and

\[ \varphi(2x, 2y) \leq 2^\beta L \varphi(x, y) \] (2.73)

for all \( x, y \in E \). Then, there exists an unique additive function \( a: E \rightarrow F \) such that

\[ \|f(x) - a(x) - g(0) - h(0)\|_\beta \leq \frac{2}{2^\beta} \frac{1}{1 - L} \chi(x, x) + \frac{1}{2^\beta} \frac{1}{1 - L} \psi(x, x), \] (2.74)
\[ \| g(x) - a(x) - g(0) \|_\beta \leq \varphi(x, 0) + \frac{2 \beta}{1 - \lambda} \chi(x, x) + \frac{1}{2 \beta} \frac{1}{1 - \lambda} \psi(x, x) \quad (2.75) \]

and

\[ \| h(x) - a(x) - h(0) \|_\beta \leq \frac{1}{2 \beta} \frac{1}{1 - \lambda} \psi(x, x) + \varphi(0, x) \quad (2.76) \]

for all \( x \in E \), where

\[ \chi(x, y) = \varphi(0, y) + \varphi(x, y) + \varphi(x, 0) + \varphi(0, y) + [\varphi(x, y) + \varphi(x, 0)] \]

and

\[ \psi(x, y) = \varphi(0, y) + [\varphi(x, y) + \varphi(x, 0)] \]

References


